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A NOTE ON INDUCED CYCLES IN KNESER GRAPHS

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Let $g(n,r)$ be the maximal order of an induced cycle in the Kneser graph $\mathrm{Kn}([n],r)$, whose vertices are the r-sets of $[n]=\{1,\dots,n\}$ and whose adjacency relation is disjointness. Thus $g(n,r)$ is the largest m for which there is a sequence $A_1,A_2,\dots,A_m\subset[n]$ of r-sets with $A_i\cap A_j=\emptyset$ if and only if $|i-j|=1$ or $m-1$. We prove that there is an absolute constant $c>0$ for which

$$g(2r+1,r)>c(2.587)^r,$$

improving previous results. Our lower bound also shows that the clique covering number of the complement of an n-cycle is at most $1.459\log_2 n$ for large enough n. Related problems concerning the order of induced subgraphs of bounded degree of Kneser graphs are discussed.

1. Introduction

Given an integer $r\geq 1$ and a set X, we define the *r-Kneser Graph* on the *ground set* X, denoted $\mathrm{Kn}(X,r)$, to be the graph whose vertices are the r-subsets of X, and whose adjacency relation is disjointness. For instance, if $|X|=5$ and $r=2$, then $\mathrm{Kn}(X,r)$ is the complement of the line graph of the complete graph on 5 vertices, *i.e.* the Petersen graph. These graphs are very natural objects and they have attracted much attention owing to a conjecture of Kneser [14], who gave a plausible value for their chromatic number. Lovász [15], using some algebraic topology, settled the then 23-year-old problem of Kneser in 1978, and Bárány [4] gave an elegant shorter proof soon afterwards. Some other references concerning Kneser graphs are [6], [10] and [16].

In this note we shall study the induced paths and cycles of Kneser graphs. For integers n and $r\geq 1$, let us denote by $g(n,r)$ the maximum order of an induced cycle of $\mathrm{Kn}([n],r)$, where as usual $[n]=\{1,\dots,n\}$. Also, let us set $g(r)$ to be the corresponding supremum for $\mathrm{Kn}(\mathbb{N},r)$. Clearly $g(n,r)$ is non-decreasing in n and $g(r)=\lim_n g(n,r)$. It turns out that this limit is finite for all r, and in fact the following bounds hold.

$$2+2^r\leq g(r)=\max_n g(n,r)\leq 1+\binom{2r}{r}. \tag{1}$$

The upper bound has been observed by several authors: Alles and Poljak [1], Alon [2], de Caen, Gregory and Pullman [7] and Tuza (see [1]); in fact, it follows from a result of Frankl [9] and Kalai [13]. The lower bound was proved by Alles by an inductive argument (see [11]). Also, Theorem 3.1 in [7] essentially gives the

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necessary techniques to prove a lower bound of the form $g(r) \geq c2^r$ for some constant $c > 2$. Our main result further improves this bound; we prove that there exists a constant $c > 0$ for which

$$g(2r+1,r) > c(2.587)^r. \tag{2}$$

We would like to remark that Dr Zs. Tuza has kindly informed us that Alles and Poljak [1] have also greately improved the lower bound in (1): they have in fact shown that $g(r) = \max_n g(n,r) > 1.2(2.50)^r$.

We remark that the construction given by Alles and Poljak [1] proves $g(n,r) > 1.2(2.50)^r$ only for $n = 9r/4$, and so one needs a 'large' ground set. It is rather pleasing that we can take our ground set in (2) of size $n = 2r+1$, since that is the minimal possible value for n: note that $\mathrm{Kn}([n],r)$ is acyclic for $n \leq 2r$. Also, this 'minimality' of n easily gives us a considerable improvement on a certain result on clique covers. Let us recall some definitions. A *clique cover* of a graph G is a family of complete subgraphs of G such that any edge of G is contained in one of the graphs in the family. The minimal cardinality of such a family is the *clique covering number* of G, and is usually denoted $\mathrm{cc}(G)$. Clique coverings were introduced in [8], and have since been studied in many papers; see *e.g.* [3], [5], [7] and [12].

Let H^n be the complement of an n-cycle. It was proved by de Caen, Gregory and Pullman [7] that $\mathrm{cc}(H^n) \leq 2\log_2(n-1)+2$. Alles and Poljak's bound of $g(9r/4,r) > 1.2(2.50)^r$ improves this to $\mathrm{cc}(H^n) < 1.695\log_2 n$. Relation (2) easily implies

$$\mathrm{cc}(H^n) < 2\frac{\log_2 n}{\log_2 2.587} < 1.459\log_2 n, \tag{3}$$

for large enough n. It is in fact conjectured in [7] that $\mathrm{cc}(H^n) = (1+o(1))\log_2 n$, although Alles and Poljak's bound and (3) seem to be the first upper bounds of the form $c\log_2 n$, with $c < 2$.

We show (2) by a simple inductive construction; we remark however that our proof involves an elementary computer search. In Section 2 we give a complete proof for the somewhat weaker bound $g(2r+1,r) > c(2.154)^r$, and make some comments on the computing involved in the proof of (2). In the last two sections, we discuss some related problems.

2. The construction

Let us define $f(n,r)$ to be the maximum of the orders of induced paths in $\mathrm{Kn}([n],r)$. Since an induced cycle of length ℓ contains an induced path with $\ell - 1$ vertices, it follows that $g(n,r)$ is at most $f(n,r)+1$. The following result shows that, on the other hand, $f(2r+1,r)$ grows about as fast as $g(2r+1,r)$. We remark that one can check that $f(2r+1,r)$ is non-decreasing and that $f(n,2) = 5$ for any $n \geq 5$. Incidentally, up to permutation of the elements of the ground set, there is only one induced path of order 5 in $\mathrm{Kn}(\mathbb{N},2)$, namely, 12, 34, 15, 24, 13, where we have omitted brackets and commas when writing the 2-sets, *e.g.* $12 = \{1,2\}$, *etc.*

Lemma 1. *For* $r \geq 2$, *we have* $2+2f(2r+1,r) \leq g(2r+3,r+1) \leq 1+f(2r+3,r+1)$.

Proof. Suppose that $r \geq 2$ and that

$$P = V_1V_2\dots V_t$$

is an induced path in $\mathrm{Kn}([2r+1],r)$ of maximal order: $t = f(2r+1,r) \geq f(5,2) = 5$. Set $x = 2r+2$ and $y = 2r+3$. Considering the $2t$ sets $V_i^x = V_i \cup \{x\}$ and $V_i^y = V_i \cup \{y\}$, $i = 1,\ldots,t$, we see that $\mathrm{Kn}([2r+3],r+1)$ contains the disjoint union of two paths of order t as an induced subgraph. We now join these two paths through a vertex to obtain an induced path of order $2t+1$. Pick $a \in V_3 \setminus V_1$ and write U_1 for the $(r+1)$-set $V_2 \cup \{a\}$. Trivially, U_1 meets neither V_1^x nor V_1^y. Moreover, it certainly meets V_i^x and V_i^y for $2 \leq i \leq t$, since $V_2 \subset U_1$ meets $V_i = V_i^x \cap V_i^y$ if $i \neq 3$ and $a \in U_1 \cap V_3^x \cap V_3^y$. This gives us the induced path and we now proceed to close a cycle. Choose an arbitrary $b \in V_{t-2} \setminus V_t$ and define $U_2 = V_{t-1} \cup \{b\}$. By similar considerations to the ones concerning U_1, we see that U_2 meets neither V_t^x nor V_t^y but does meet V_i^x and V_i^y for $1 \leq i \leq t-1$. Finally, we note that $U_1 \cap U_2 \supset V_2 \cap V_{t-1} \neq \emptyset$ if $t \geq 5$, and this shows that U_2 closes the cycle as required. ∎

We shall now bound $f(2r+1,r)$ from below by giving a recursive construction for induced paths in $\mathrm{Kn}([2r+1],r)$. Our construction is given in Lemma 2 below. As usual, we denote the k-subsets of a set X by $X^{(k)}$, $k \geq 0$. For $s = 1,2,\ldots$, let us define G_s to be the bipartite graph whose vertex classes are $[2s]^{(s)}$ and $[2s]^{(s-1)}$, two vertices in different classes being adjacent if and only if they are disjoint. Define $w(s)$ to be the maximum number of vertices in $[2s]^{(s)}$ in an induced path in G_s.

Lemma 2. *For every $r \geq 2$ and $s \geq 1$,*

$$f(2(r+s)+1,r+s) \geq \begin{cases} w(s)(f(2r+1,r)+1)-1 & \text{if } f(2r+1,r) \text{ is odd} \\ w(s)f(2r+1,r)-1 & \text{if } f(2r+1,r) \text{ is even.} \end{cases}$$

Proof. Let

$$W = A_1B_1A_2B_2\ldots A_{m-1}B_{m-1}A_m$$

be an induced path in G_s, where $A_i \in [2s]^{(s)}$, $B_j \in [2s]^{(s-1)}$, $1 \leq i \leq m$, $1 \leq j \leq m-1$, $m \geq 2$, and let

$$P = V_1V_2\ldots V_t$$

be an induced path in $\mathrm{Kn}(\{2s+1,\ldots,2(r+s)+1\},r)$, where $r \geq 2$ and $t \geq 5$ is odd. We now construct an induced path in $\mathrm{Kn}([2(r+s)+1],r+s)$ of order $m(t+1)-1$.

For a set $A \subset [2s]$, write A^c for $[2s] \setminus A$. Note that, given two disjoint sets X_1, X_2, and families of r_i-sets $\mathcal{F}_i \subset X_i^{(r_i)}$, where $r_i \geq 1$ $(i = 1, 2)$, the graph induced by $\mathcal{F}_1 \vee \mathcal{F}_2 = \{F_1 \cup F_2 \subset X_1 \cup X_2 : F_i \in \mathcal{F}_i,\ i = 1, 2\}$ in $\mathrm{Kn}(X_1 \cup X_2, r_1+r_2)$ is the (categorical) product of the graphs induced by the $\mathcal{F}_i$ in the $\mathrm{Kn}(X_i, r_i)$. We start our construction by setting $\mathcal{A} = \{A_1, A_1^c, \ldots, A_m, A_m^c\}$ and noting that $\mathcal{A}$ induces a disjoint union of at least $\lceil m/2 \rceil$ edges in $\mathrm{Kn}([2s],s)$. Hence $\mathcal{A} \vee \{V_1,\ldots,V_t\}$ induces a disjoint union of at least m paths of order t in $\mathrm{Kn}([2(r+s)+1],r+s)$ and, as t is odd, the pairs of end-vertices of these paths are $\{V_1 \cup A_i, V_t \cup A_i\}$ and $\{V_1 \cup A_i^c, V_t \cup A_i^c\}$, $i = 1,\ldots,m$. Set P_i to be the path with end-vertices $\{V_1 \cup A_i, V_t \cup A_i\}$, $i = 1,\ldots,m$, and let $\mathcal{P} \subset [2(r+s)+1]^{(r+s)}$ be the set of mt vertices used by these m paths. We shall, using a method analogous to the one in the proof of Lemma 1, join these P_i together to obtain our path in $\mathrm{Kn}([2(r+s)+1],r+s)$.

Pick $a \in V_3 \setminus V_1$ and $b \in V_{t-2} \setminus V_t$. For $i = 1,\ldots,m-1$, define

$$U_i = \begin{cases} V_{t-1} \cup B_i \cup \{b\} & \text{if } i \text{ is odd} \\ V_2 \cup B_i \cup \{a\} & \text{if } i \text{ is even.} \end{cases}$$

Note that these U_i span an independent set. Indeed, for $1 \leq i < j \leq m-1$, we have $U_i \cap U_j \supset V_2 \cap V_{t-1} \neq \emptyset$, as $t \geq 5$. Furthermore, for $1 \leq i \leq m-1$, we claim that U_i is adjacent to exactly two vertices in $\mathcal{P}$, namely, $V_t \cup A_i$ and $V_t \cup A_{i+1}$ if i is odd and $V_1 \cup A_i$ and $V_1 \cup A_{i+1}$ if i is even. We check this statement only for even i, since the other case is analogous. Let Y be a vertex in $\mathcal{P}$, $Y = V_j \cup W$, where W is either A_k or A_k^c for some $1 \leq k \leq m$. First of all, $U_i \supset V_2$ meets $Y \supset V_j$ if $4 \leq j \leq t$ or $j = 2$, since $V_2 \cap V_j \neq \emptyset$ in that case. Secondly, $a \in U_i \cap Y \neq \emptyset$ if $j = 3$. Finally, if $j = 1$ then, noting that $U_i \cap Y = B_i \cap A_k$, we have that U_i does not meet $Y = V_1 \cup A_k$ if and only if $k = i$ or $k = i + 1$. ∎

Corollary 3. *There is an absolute constant $c > 0$ for which $g(2r+1, r) > c10^{r/3} > c(2.154)^r$ holds for every $r \geq 1$.*

Proof. In view of Lemma 1, it is enough to prove that $f(2r+1, r) > c_0 10^{r/3}$ for some constant $c_0 > 0$. Let us apply induction on r in order to bound $f(2r+1, r)$ from below. Recall that $f(5,2) = 5$. By Lemma 1, we have that $f(7,3) \geq 11$ and $f(9,4) \geq 23$. Also,

$$W = 123, 56, 124, 36, 125, 34, 156, 24, 135, 26, 345, 16, 235, 14, 236, 15, 246, 13, 456$$

is an induced path in G_3, so that $w(3) \geq 10$. Thus Lemma 2 yields, for $u \geq 0$,

$$f(6u+5, 3u+2) = f(2(3u+2)+1, 3u+2) \geq 6 \cdot 10^u - 1,$$
$$f(6u+7, 3u+3) = f(2(3u+3)+1, 3u+3) \geq 12 \cdot 10^u - 1$$

and

$$f(6u+9, 3u+4) = f(2(3u+4)+1, 3u+4) \geq 24 \cdot 10^u - 1,$$

which implies that indeed $f(2r+1, r) > c_0 10^{r/3}$ for a positive c_0. ∎

Theorem 4. *There is an absolute constant $c > 0$ for which $g(2r+1, r) > c300^{r/6} > c(2.587)^r$ holds for every $r \geq 1$.*

It is clear from the proof of Corollary 3 that we need only show that $w(6) \geq 300$ in order to prove the lower bound in Theorem 4. Unfortunately, however, the only way we have managed to prove such a bound for $w(6)$ is by a computer search. We do not describe the induced path we have found; we only remark that an elementary depth-first algorithm finds such a path after hitting dead-ends only 66 times, *i.e.* the 67th maximal path we analyse has 300 vertices in $[12]^{(6)}$.

Our algorithm explores the possible extensions of the currently considered path in a certain natural order. More precisely, let us assume that we have an induced path in G_s ending at an $(s-1)$-set and we are trying to consider all its possible extensions for further search. Recall that, for $k = 1, 2, \ldots$, the *colex order* on $\mathbb{N}^{(k)}$ is the ordering in which A precedes B if and only if $\sup A \triangle B \in B$, where $\triangle$ denotes symmetric difference. The order in which we make our search is simply the colex order on $[2s]^{(s)}$, *i.e.* we search the smallest possible extension in the colex order first. If, on the other hand, our path ends at an s-set, our search follows the *opposite* of the colex order on $[2s]^{(s-1)}$.

It is very likely that by using better algorithms and more computer time one could somewhat improve the lower bound in Theorem 4. However, to prove $f(r) >$

$(4 - o(1))^r$ one would have to show that $\sup w(s)^{1/s} = 4$. Therefore, in order to make real progress, more systematic work needs to be done on estimating the order of $w(s)$.

3. General induced subgraphs

In this section we discuss a related problem concerning induced subgraphs of Kneser graphs. For a graph H, let us denote its order by $|H|$ and its maximal and minimal degrees by $\Delta(H)$ and $\delta(H)$, respectively. Also, let us set

$$h(\Delta, r) = \max\{|H| : H \text{ an induced subgraph of } \mathrm{Kn}(\mathbf{N}, r),\ \delta(H) \geq 1,\ \Delta(H) \leq \Delta\}.$$

Our next result estimates $h(\Delta, r)$ from both sides; the bounds we give are reasonably strong: their quotient is smaller than 4. The upper bound is proved by the linear algebraic method of Frankl [9] and Kalai [13] (see also Alon [2]). It should be noted that, using a graph-theoretic lemma in [3], one can prove the upper bound below by quoting the set-pair system result of Frankl [9] and Kalai [13]. However, our method is more direct.

Theorem 5. *For* r, $\Delta \geq 1$,

$$\max\left\{\binom{2r}{r}, (\Delta+1)\binom{2r-2}{r-1}\right\} \leq h(\Delta, r) \leq \Delta\binom{2r}{r}.$$

Proof. Let r, $\Delta \geq 1$ be fixed. We first prove the lower bound. Note that $\mathrm{Kn}(\mathbf{N}, r)$ contains as an induced subgraph the disjoint union of $\binom{2r}{r}/2$ edges; indeed, simply consider $\mathrm{Kn}([2r], r) \subset \mathrm{Kn}(\mathbf{N}, r)$. Hence $h(\Delta, r) \geq \binom{2r}{r}$. Moreover, let us write $n = 2r + \Delta - 1$ and define

$$\mathcal{F} = \{F \in [n]^{(r)} : |F \cap [\Delta+1]| = 1\}.$$

Let $K^{\Delta+1,\Delta+1}$ denote the balanced complete bipartite graph on $2(\Delta+1)$ vertices, and let K' denote the graph obtained from $K^{\Delta+1,\Delta+1}$ by the deletion of a perfect matching. It is easily verified that $\mathcal{F}$ induces a disjoint union of $\binom{2r-2}{r-1}/2$ copies of K'. Thus $h(\Delta, r) \geq (\Delta+1)\binom{2r-2}{r-1}$, which completes the proof of the lower bound.

Let us now turn to the upper bound. Let $H \subset \mathrm{Kn}(\mathbf{N}, r)$ be a finite induced subgraph of $\mathrm{Kn}(\mathbf{N}, r)$ with vertices $v_1, \ldots, v_m$, $m = |H|$, none of them isolated. Let $\{x_i \in \mathbf{R}^{2r} : i \geq 1\}$ be a set of points in $\mathbf{R}^{2r}$ such that for all $i_1 < \cdots < i_{2r}$ the vectors $x_{i_1}, \ldots, x_{i_{2r}}$ are linearly independent, and hence $x_{i_1} \wedge \cdots \wedge x_{i_{2r}} \in \bigwedge^{2r} \mathbf{R}^{2r}$ is non-zero. For each $1 \leq i \leq m$, set $y_i = \bigwedge_{j \in v_i} x_j \in \bigwedge^r \mathbf{R}^{2r}$. Let $\{e_i : 1 \leq i \leq m\}$ be a fixed basis of $\mathbf{R}^m$ and define a linear map $L : \mathbf{R}^m \to \bigwedge^r \mathbf{R}^{2r}$ by sending e_i to y_i, $1 \leq i \leq m$. Also, let $\phi : \bigwedge^{2r} \mathbf{R}^{2r} \to \mathbf{R}$ be an arbitrary isomorphism and set $\lambda_{ij} = \phi(y_i \wedge y_j)$, $1 \leq i, j \leq m$. Denote the m by m matrix (λ_{ij}) by Λ.

Note that $\ker L \subset \ker \Lambda$, where Λ is regarded as a linear map on $\mathbf{R}^m$ acting on the column vectors of $\mathbf{R}^m$ by left-multiplication. Indeed, let $(\alpha_j)_1^m \in \mathbf{R}^m$ be such that

$$\sum_{j=1}^{m} \alpha_j y_j = L\left[\sum_{j=1}^{m} \alpha_j e_j\right] = 0.$$

Then, for each $1 \leq i \leq m$, applying $y_i \wedge -$ followed by ϕ, we get $\sum_{j=1}^{m} \alpha_j \lambda_{ij} = 0$. Hence $\Lambda(\alpha_j) = 0$. Denote the rank of Λ by $\rho(\Lambda)$. The following completes the proof.

Claim. $m/\Delta \leq \rho(\Lambda) \leq \binom{2r}{r}$.

Note that

$$\begin{aligned} m &= \dim\ker L + \dim \operatorname{im} L \\ &\leq \dim\ker \Lambda + \binom{2r}{r} \\ &= m - \rho(\Lambda) + \binom{2r}{r}, \end{aligned}$$

which gives the upper bound for $\rho(\Lambda)$. The lower bound follows easily from the fact that, in Λ, each row has at least one and each column at most Δ non-zero entries. Indeed, if we assume that the first $\rho = \rho(\Lambda)$ columns of Λ span the image of Λ, then every row of Λ must have one of its non-zero entries in those columns and, therefore, the number of rows m of Λ is at most $\Delta\rho$, since each column has at most Δ non-zero entries. ∎

4. Concluding remarks and open problems

By using the set-pair system result of Frankl [9] and Kalai [13], one can show that $w(s) \leq 1 + \binom{2s-1}{s-1}$, $s \geq 1$. Hence the best lower bound for $g(2r+1, r)$ one can possibly get by our construction (with s fixed) seems to be exponentially smaller than $\binom{2r}{r}$. Being unable to decide whether $g(r) = o\left[\binom{2r}{r}\right]$ holds, we were led to the following weakening of that question.

Problem 6. *Are there absolute constants $\Delta_0 \geq 1$ and $\epsilon_0 > 0$ such that for every $r \geq 1$ the graph* $\mathrm{Kn}([2r+1], r)$ *contains a connected induced subgraph H_r with $\Delta(H_r) \leq \Delta_0$ and $|H_r| \geq \epsilon_0\binom{2r+1}{r}$?*

Alles and Poljak [1] have raised the following very interesting problem. Let $n = n(r)$ be the smallest integer for which $g(n, r) = g(r)$. The question then is to determine or estimate $n(r)$. It has not been disproved that $n(r) = 2r + 1$.

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